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# Mathematical Analysis of P-Stability Maps for Parametric Conic Vector Optimization

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**Abstract**

Stability analysis for nonlinear programming systems deals with the possible changes of the system parameters and/or equations that maintain the stability of the solutions. It is a crucial requirement to study the nonlinear system and its practical values, specifically the economic impact in most real-world applications. This paper presents some outcomes in connection with stability analysis corresponding to parametric conic vector optimization problems. For these last optimization problems, two novel types of P-Stability maps, which are the P-Stability notion map and the P-Stability perturbation map, are considered based on six kinds of sets: P-feasible set, P-solvability set, the first, second, third, and fourth kinds of P-Stability notion sets with respect to a specific domination cone P. Furthermore, qualitative characteristics of the P-Stability maps under some continuity and convexity assumptions on the objective function are provided and proved. Specifically, the connections between the P-Stability maps and the P-Stability notion set are investigated. Accordingly, these characteristics were extended to the P-perturbation maps. In addition, the idea of P-stability has heavily used in different applications like network privacy, engineering fields, and some business financial models.

*Keywords Parametric Vector Optimization Problems (PVOP); Domination Cone; Perturbation Maps; Set-valued Maps; Stability Notions*

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## 1. Introduction:

Studies related to stability and sensitivity analysis for optimization problems are necessary not just from a theoretical but also from a practical point of view. This follows from the numerous applications of optimization theory within different fields. Extensive research was done on the stability and sensitivity analysis of PVOP [3, 6–8, 10, 22, 33]. Stability is commonly related to qualitative analysis. In this paper, for stability, we study various continuity properties for the stability maps associated with parametric vector optimization problems. Sensitivity, on the other hand, is related to differential stability and is relevant to the quantitative analysis. It is fulfilled, in general, by the study of the derivative and sub-differential expressions of the map under consideration.

Many results for stability and sensitivity analysis have been stated and proven within vector optimization theory and applications [5, 9, 19–21, 34,35]. Tanino [31, 32] has acquired some results related to sensitivity analysis for vector optimization problems (VOP) since several decades, based on the principle of contingent derivatives for set-valued maps pioneered by Aubin [2]. Then, in 1996, Kuk et al. expanded these results [18]. Notably, Shi investigated various quantitative results for the perturbation map associated with VOP, under some convexity assumptions [29, 30]. The existence of these assumptions leads to weaker results in comparison to the general case. This approach is referred to as the first approach or the Japanese one.

Simultaneously, the second approach, which is known as the Egyptian one, was presented by Osman in [24]. In this last proceeding, stability notions for single objective decision-making problems have been discussed using a parametric approach. Existing research on the qualitative analysis of fundamental notions in the parametric multi-objective convex programming problem, where the parameter exists in either the objective or in the constraint function, can be found in [24, 25]. Later, the stability notions for parametric optimization problems with parameters in both the objective and constraint functions were reinterpreted and qualitatively examined for multi-objective convex programming problems with their various applications, e.g. [10, 11, 16].

In the previous decay, it was noticed that the idea of P-stability has heavily used in network privacy, different engineering fields and some business applications. It may be used to create a complete polynomial randomized approximation system for graph masking and measurement risk in the context of online social networks [13,26].

This paper tackles the parametric conic vector optimization problems (PCVOP), where the optimization is restricted to being over a domination cone and is structured as follows. First, in the subsequent section, the basic problem is presented. Next, in Section 3, new types of P-Stability notion maps based on six kinds of sets, for PCVOP, are defined, where P denotes a cone. Moreover, qualitative properties are displayed and proved. Then in Section 4, the definition of the P-Stability perturbation map is extended, and some of its characteristics are shown. An illustrative example is given in Section 5 before Section 6 concludes the paper.

## 2. Basic Problem

In this study, we investigate a parametric conic vector optimization problem $\left(E_P(u)\right)$, for $u \in U$ a perturbation vector parameter in $\mathbb{R}^m$,

$$E_P(u) \to \begin{cases} P - \min\limits_{x \in X(u)} f(x,u) = \left(f_1(x,u), f_2(x,u), \dots, f_p(x,u)\right) \\ \text{subject to} \\ X(u) = \{x \in \Re^n : g_i(x,u) \le 0, i = 1, \dots, q\} \subset \mathbb{R}^n \end{cases}$$

where $f$ is a $p$-dimensional objective function on $\mathbb{R}^n \times \mathbb{R}^m$, $g$ is a $q$-dimensional constraint function on $\mathbb{R}^n \times \mathbb{R}^m$, and $X$ identifies what is known by feasible decision set, in fact it is a set-valued map from $\Re^m$ to $\Re^n$, and $P$ represents a nonempty pointed closed convex cone in $\mathbb{R}^p$ that acts as the objective space's domination cone. [23].

On the real-vector space $\mathbb{R}^m$, define the set-valued map $Y$ [2] by

$$\mathrm{Y(u) = f(X(u), u) = \{y \in \mathbb{R}^p : y = f(x, u) \text{ for some } x \in X(u)\}}$$

In the objective space, this map is regarded as the feasible set map. In order to define a solution to the problem $(E_P(u))$, a partial order is induced by the closed pointed with nonempty interior convex cone $P$ in the objective space $\mathbb{R}^p$ is considered. If $P$ is pointed, then it contends that

$$l(P) = P \cap (-P) = \{0\}.$$

Consequently, for $y, \tilde{y} \in \mathbb{R}^p$, we define the following partial orders

$y \leqq_P \bar{y}$ iff $y - \bar{y} \in P \setminus l(P) = P \setminus \{0\}$,
$y <_P \bar{y}$ iff $y - \bar{y} \in int(P)$.

The P-minimal points of a set are defined as follows in compliance with these orders:

- **Definition 2.1 (P-minimal point)**

For any set $S \subset \mathbb{R}^p$, $\hat{y} \in S$ is called a P −minimal point of S with respect to P, if there is no $y \in S$ s.t. $y \leqq_P \hat{y}$. Consequently, $Min_P S$ represents the set of all P-minimal points of S.
The "P-Minimization" in the above problem $(E_P(u))$ is equivalent to obtaining the set $Min_P Y(u)$ see e.g., [12]. In view of this solution concept, the following set-valued map Φ is defined

$$\Phi: \mathbb{R}^m \to \mathcal{P}(\mathbb{R}^p)$$
$$u \to \Phi(u) = Min_P Y(u),$$

where $\mathcal{P}(\mathbb{R}^p)$ is the power set of $\mathbb{R}^p$, and is called the perturbation map. Tanino did, in fact, investigate the quantitative outcomes of this map's behavior [31, 32] which then was improved by Shi [29, 30].

We end this section by stating some definitions regarding point-to-set maps which are serviceable through the paper.

- **Definition 2.2 (Compact set-valued map)**

Consider A and B to be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and let $F: A \to \mathcal{P}(B)$. The set-valued map F is uniformly compact near a point $x_0 \in A$, if $x_0$ has an open neighborhood $\mathcal{N}$ where $\overline{\bigcup_{x \in \mathcal{N}(x_0)} F(x)}$ is compact. F is called uniformly compact if it is uniformly compact near x for all $x \in A$.

- **Definition 2.3 (Upper semicontinuous set-valued map)** [1]

Consider A and B to be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and let $F: A \to \mathcal{P}(B)$. The set-valued map F is called sequentially upper semicontinuous (S.u.s.c) at a point $x_0 \in X$, if for each sequence $\{x_n\} \subset A$, converging to $x_0$, and every sequence $y_n \in F(x_n)$, we have $dist(y_n, F(x_0)) \to 0, n \to \infty$. F is called S.u.s.c if it is S.u.s.c at x for all $x \in A$.

- **Definition 2.4 (Closed set-valued map)**

Consider A and B to be subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and let $F: A \to \mathcal{P}(B)$. F is closed if and only if $\overline{F(V)} \subseteq F(\overline{V})$ for any $V \subset A$.

## 3. P-Stability Notion Sets and Maps

The current section is devoted to studying the stability of a certain efficient solution lying inside the domination cone (which is normally determined by decision-makers) against all possible variations in the relevant vector parameter and establishing the stability notions that include all vector parameter values that keep this specific solution within the efficient set or keep the feasible points converging to the efficient solution within the cone. These stability notions are called P-stability sets. Therefore, the new definitions of P-stability notion maps for the PCVOP are presented in the following series of definitions.

- **Definition 3.1 (The feasible set of parameters)**

The feasible set of parameters, denoted by $F_s$, for the parametric vector optimization problem $E_P$ is defined by

$$F_S = \{u \in \mathbb{R}^m : X(u) \neq \emptyset\}.$$

- **Definition 3.2 (The P-solvability set)**

The P-solvability set, denoted by $S_S^P$, for the problem $E_P$ is defined by

$$S_S^P = \{u \in \mathbb{R}^m : Min_P Y(u) \neq \emptyset\}.$$

- **Definition 3.3 (The first kind P-Stability notion map)**

The first kind P-stability notion map, denoted by $S_1^P$, is defined as the set-valued map

$$\mathbb{R}^n \to \mathcal{P}(\mathbb{R}^m)$$
$$\bar{x} \to S_1^P(\bar{x}) = \{u \in S_s^P : \bar{y} = f(\bar{x}, u) \in Min_P Y(u)\}.$$

For a given efficient solution $\bar{x} \in \Re^n$, $S_1^P(\bar{x})$ is named the first kind P-Stability set.

- **Definition 3.4 (The second kind P-Stability notion map $S_2^P$)**

Let $\bar{u} \in S_S^P$ with associated efficient solution $\bar{x}$ so that $f(\bar{x}, \bar{u}) \in Min_P Y(\bar{u})$, and $\mathfrak{I}(I)$ denotes <u>either</u> the unique side of the feasible decision set $X(\bar{u})$ which contains $\bar{x}$ <u>or</u> $int(X(\bar{u}))$, where $I = \{1,2, \dots, l\} \subset \{1,2, \dots, q\}$, thus

$$\mathfrak{I}(I) = \{x \in \mathbb{R}^n: \ g_i(x, \bar{u}) = 0; \ \text{if } i \in I \text{ and } g_i(x, \bar{u}) < 0; \ \text{if } i \notin I\}$$

The second kind P-Stability notion map $S_2^P$ is defined as the set-valued map by

$\mathbb{R}^n \to \mathcal{P}(\mathbb{R}^m)$

$$\bar{x} \to S_2^P(\bar{x}, I) = \{u \in S_S^P: \mathrm{Min}_P Y(\bar{u}) \cap \mathfrak{I}(I) \neq \emptyset\}.$$

The set $S_2^P(\bar{x}, I)$, for a feasible solution $\bar{x}$, is said to be the second kind P-Stability set

- **Definition 3.5 (The third kind P-Stability notion map $S_3^P$)**

Assume that $\bar{u} \in S_S^P$ with an efficient solution $\bar{x}$ where $f(\bar{x}, \bar{u}) \in \mathrm{Min}_P Y(\bar{u})$, $x^*$ is a feasible point in $X(\bar{u})$ and $\delta > 0$. Then, the third kind P-Stability notion map $S_3^P$ is the set-valued map defined over $\mathbb{R}^n \times \mathbb{R}^n \times [0, \infty[$ into $\mathbb{R}^m$ by

$$S_3^P(\bar{x}, x^*, \delta) = \{u \in S_S^P: \|f(x^*, u) - f(\bar{x}, \bar{u})\| < \delta, f(\bar{x}, u) \in \mathrm{Min}_P Y(u)\}.$$

For given $\bar{x}, x^*$and $\delta$, the set $S_3^P(\bar{x}, x^*, \delta)$ denotes the third kind P-Stability set.

- **Definition 3.6 (The fourth kind P-Stability notion map $S_4^P$)**

For an efficient solution $\bar{x}$ with $\bar{u} \in S_S^P$ such that $f(\bar{x}, \bar{u}) \in \mathrm{Min}_P Y(\bar{u})$ and $\delta > 0$, the fourth kind P-Stability set-valued map $S_4^P$ is defined by,

$$S_4^P(\bar{x}, \delta) = \{u \in S_S^P: \exists x \in X(\bar{u}), \|f(x, u) - f(\bar{x}, \bar{u})\| < \delta, f(\bar{x}, u) \in \mathrm{Min}_P Y(u)\}.$$

The set $S_4^P(\bar{x}, \delta)$, for a given $\bar{x}$ and $\delta$, denotes the fourth kind P-Stability set.

It is to be noted that, the first kind P-stability set-valued map at a given efficient solution $\bar{x}$, with associated parameter $\bar{u}$, collects all the parameters which will keep $\bar{x}$ a minimizer. The second kind P-stability map is linked with the constraint function and collects those parameters which reserve the fact that $\bar{x}$ is a minimizer. However, the third and fourth kinds collect all the parameters for which the objective function at those parameters lies in the neighborhood of the optimum value $f(\bar{x}, \bar{u})$. In subsequent some mathematical properties for the $P$-stability sets $S_i^P(\bar{x})$ and P-Stability notion maps $S_i^P$ $(i = 1,2,3,4)$ are stated and proved.

- **Proposition 3.1 (Uniqueness of $S_1^P(\bar{x})$)**

If f is strictly convex on $\mathbb{R}^n \times \mathbb{R}^m$, and $\bar{x}_1, \bar{x}_2$ are different efficient solutions, then

$$S_1^P(\bar{x}_1) \cap S_1^P(\bar{x}_2) = \emptyset.$$

**Proof.** By contradiction, assume that $S_1^P(\bar{x}_1) \cap S_1^P(\bar{x}_2) \neq \emptyset$, then $\exists u_0 \in S_1^P(\bar{x}_1) \cap S_1^P(\bar{x}_2)$, i.e., $u_0$ corresponds to an efficient solution $\bar{x}_1$ and $\bar{x}_2$. This is in opposition to the fact that $f$ being strictly convex. So, the proposition is proved.

- **Proposition 3.2 (Uniform compactness of $S_1^P$)**

If f is a convex function on $\mathbb{R}^n \times \mathbb{R}^m$ and the map $\Phi = \mathrm{Min}_P Y(.)$ is closed, then the P-Stability notion map $S_1^P$ is uniformly compact.

**Proof**. It is enough to proof that $S_1^P(\bar{x})$ is closed, for any efficient solution $\bar{x} \in \mathbb{R}^n$. Let $\{u_k\}$ be a sequence in $S_1^P(\bar{x})$ and converging to $u_0$ in $\mathbb{R}^m$. It is sufficient to show that $u_0 \in S_1^P(\bar{x})$. We have $\bar{y}_k = f(\bar{x}, u_k) \in \mathrm{Min}_P Y(u_k)$. In fact, f is continuous, as it is convex function in [27], then

$$\lim_{k\to\infty} f(\bar{x}, u_k) = f\left(\bar{x}, \lim_{k\to\infty} u_k\right) = f(\bar{x}, u_0)$$

Since $\Phi = Min_P Y(.)$ is closed, then $f(\bar{x}, u_0) \in Min_P Y(u_0)$, this means that $u_0 \in S_1^P(\bar{x})$. Thus $S_1^P(\bar{x})$ is closed, and therefore $S_1^P$ is uniformly compact at $\bar{x}$. So, the proposition is proved.

- **Proposition 3.3 (Upper semi-continuity of $S_1^P$)**

The P-Stability notion map $S_1^P$ is sequentially upper semicontinuous at any efficient solution $\bar{x} \in \mathbb{R}^n$, if f is a convex function on $\mathbb{R}^n \times \mathbb{R}^m$and $\Phi = \mathrm{Min}_P Y(.)$ is closed.

**Proof.** Let $\{\bar{x}_k\} \subset \mathbb{R}^n$, such that $\bar{x}_k \to \bar{x}$, $u_k \in S_1^P(\bar{x}_k)$, and $u_k \to u_0$, this is valid following the fact that $S_1^P$is uniformly compact. To prove the proposition, it is enough to show that $u_0 \in S_1^P(\bar{x})$. Since $u_k \in S_1^P(\bar{x}_k)$, this means that $\bar{y}_k = f(\bar{x}_k, u_k) \in \mathrm{Min}_P Y(u_k)$. In fact, f is continuous, since f is convex. So

$$\lim_{k\to\infty} \bar{y}_k = \lim_{k\to\infty} f(\bar{x}_k, u_k) = f\left(\lim_{k\to\infty} \bar{x}_k, \lim_{k\to\infty} u_k\right) = f(\bar{x}, u_0).$$

Hence, $f(\bar{x}, u_0) \in Min_P Y(u_0)$, because $Min_P Y(.)$ is closed, this means that $u_0 \in S_1^P(\bar{x})$. So, the proposition is proved.

The uniqueness of the second kind stability set is proved in an analogous manner as Proposition 3.1. Following are some properties that are substantial for the P-Stability set $S_3^P(\bar{x}, x^*, \delta)$ and P-Stability notion map $S_3^P$.

- **Proposition 3.4 (Uniqueness of $S_3^P(\bar{x}, x^*, \delta)$)**

If f is one to one function, $x_1^*$ and $x_2^*$ are two distinct feasible points in $X(\bar{u})$, and $\delta_1, \delta_2 > 0$ are arbitrary numbers, such that $S_3^P(\bar{x}, x_1^*, \delta_1) \neq S_3^P(\bar{x}, x_2^*, \delta_2)$.
then

$$S_3^P(\bar{x}, x_1^*, \delta_1) \cap S_3^P(\bar{x}, x_2^*, \delta_2) = \emptyset.$$

Moreover, if $\bar{x}_1$ and $\bar{x}_2$ are two distinct efficient solutions, then for any feasible point $x^*$,

$$S_3^P(\bar{x}_1, x^*, \delta_1) \cap S_3^P(\bar{x}_2, x^*, \delta_2) = \emptyset.$$

**Proof.** Suppose the contrary, i.e., $\exists u_0 \in S_3^P(\bar{x}, x_1^*, \delta_1) \cap S_3^P(\bar{x}, x_2^*, \delta_2)$. Following the definition of $S_3^P(\bar{x}, x^*, \delta)$ we get that $f(\bar{x}, u_0) \in Min_P Y(u_0)$ and

$$\|f(x_1^*, u_0) - f(\bar{x}, \bar{u})\| < \delta_1, \quad (1)$$
$$\|f(x_2^*, u_0) - f(\bar{x}, \bar{u})\| < \delta_2, \quad (2)$$

Inequalities (1) and (2) implies that

$$\|f(x_1^*, u_0) - f(x_2^*, u_0)\| \le \|f(x_1^*, u_0) - f(\bar{x}, \bar{u})\| + \|f(x_2^*, u_0) - f(\bar{x}, \bar{u})\| < \delta_1 + \delta_2 < 2\delta.$$

where $\delta = max(\delta_1, \delta_2)$ and is an arbitrary positive number. Hence, $f(x_1^*, u_0) = f(x_2^*, u_0)$ which leads to contradiction with the assumption that $f$ is one-to-one. In an analogous manner, the second equality can be proved.

- **Proposition3.5 (Uniform compactness of P-Stability notion map $S_3^P$)**

If f is a convex function on $\mathbb{R}^n \times \mathbb{R}^m$ and the map $\Phi = Min_P Y(.)$ is closed, then the P-Stability notion map $S_3^P$ is uniformly compact (locally bounded) at any efficient solution $\bar{x} \in \mathbb{R}^n$.

**Proof**. To prove the proposition, it is enough to show that $S_3^P(\bar{x}, x^*, \delta)$ is closed. Consider a sequence $\{u_k\} \xrightarrow[k\to\infty]{} u_0$, where $\{u_k\}$ in $S_3^P(\bar{x}, x^*, \delta)$ i.e.

$$\|f(x^*, u_k) - f(\bar{x}, \bar{u})\| \le \delta - \epsilon, \quad (3)$$

where $\epsilon \ll 1$, and $f(\bar{x}, u_k) \in Min_P Y(u_k)$. Since $Min_P Y(.)$ is closed and $f$ is convex, then $f(\bar{x}, u_0) \in Min_P Y(u_0)$ and

$$\lim_{k\to\infty} f(x^*, u_k) = f(x^*, u_0).$$

So, for $k \gg 1$

$$\|f(x^*, u_k) - f(x^*, u_0)\| < \epsilon, \quad (4)$$

Inequalities (3) and (4) implies that

$$\|f(x^*, u_0) - f(\bar{x}, \bar{u})\| \le \|f(x^*, u_k) - f(x^*, u_0)\| + \|f(x^*, u_k) - f(\bar{x}, \bar{u})\| < \delta.$$

Then $u_0 \in S_3^P(\bar{x}, x^*, \delta)$, which proves the proposition.

- **Proposition 3.6 (Upper semi-continuity of P-Stability notion map $S_3^P$)**

The P-Stability notion map $S_3^P$ is sequentially upper semi-continuous on any efficient solution $\bar{x} \in \mathbb{R}^n$, if f is a convex function on $\mathbb{R}^n \times \mathbb{R}^m$ and $\Phi = Min_P Y(.)$ is closed

**Proof.** Following the compactness of $S_3^P$, let $\{\bar{x}_k\} \subset \mathbb{R}^n$ be a sequence of feasible solutions, such that $\bar{x}_k \to \bar{x}$ and $u_k \in S_3^P(\bar{x}_k, x^*, \delta)$ such that $u_k \to u_0$.

To prove the proposition, we must show that $u_0 \in S_3^P(\bar{x}, x^*, \delta)$. Since $u_k \in S_3^P(\bar{x}_k, x^*, \delta)$, then $\bar{y}_k = f(\bar{x}, u_k) \in Min_P Y(u_k)$ and $\|f(x^*, u_k) - f(\bar{x}, \bar{u})\| < \delta$. Moreover, since $f$ is a convex function on $\mathbb{R}^n \times \mathbb{R}^m$, this implies that $f$ is continuous, this leads to

$$\lim_{k\to\infty} \bar{y}_k = \lim_{k\to\infty} f(\bar{x}, u_k) = f\left(\bar{x}, \lim_{k\to\infty} u_k\right) = f(\bar{x}, u_0).$$

But $Min_P Y(.)$ is closed, so $f(\bar{x}, u_0) \in Min_P Y(u_0)$. Moreover,

$$\lim_{k\to\infty} f(x^*, u_k) = f\left(x^*, \lim_{k\to\infty} u_k\right) = f(x^*, u_0).$$

Therefore, for $k \gg 1$

$$\|f(x^*, u_0) - f(\bar{x}, \bar{u})\| \le \|f(x^*, u_0) - f(x^*, u_k)\| + \|f(x^*, u_k) - f(\bar{x}, \bar{u})\| < \delta.$$

We have

$$\|f(x^*, u_0) - f(\bar{x}, \bar{u})\| < \delta,$$
$$f(\bar{x}, u_0) \in Min_P Y(u_0).$$

Consequently,

$$u_0 \in S_3^P(\bar{x}, x^*, \delta).$$

Thus, the proposition is proved.

Thereafter, some properties for the P-Stability set $S_4^P(\bar{x}, \delta)$ and P-Stability notion map $S_4^P$ are stated and proved.

- **Proposition 3.7 (Uniqueness of $S_4^P(\bar{x}, \delta)$)**

Let f be one-to-one function and $\delta_1, \delta_2$ two arbitrary distinct positive numbers such that $S_4^P(\bar{x}, \delta_1) \ne S_4^P(\bar{x}, \delta_2)$. Then,

$$S_4^P(\bar{x}, \delta_1) \cap S_4^P(\bar{x}, \delta_2) = \emptyset.$$

***Proof.*** If there exists $u_0 \in S_4^P(\bar{x}, \delta_1) \cap S_4^P(\bar{x}, \delta_2)$ (i.e., assume the contrary). Then, from the definition of $S_4^P(\bar{x}, \delta)$, we get $f(\bar{x}, u_0) \in Min_P Y(u_0)$ and

$$\exists x_1 \in X(\bar{u}), \quad \|f(x_1, u_0) - f(\bar{x}, \bar{u})\| < \delta_1, \quad (5)$$
$$\exists x_2 \in X(\bar{u}), \quad \|f(x_2, u_0) - f(\bar{x}, \bar{u})\| < \delta_2, \quad (6)$$

There are two cases for $x_1$ and $x_2$

Case 1. If $x_1 \ne x_2$, then from (5) and (6) we get

$$\|f(x_1, u_0) - f(x_2, u_0)\| \le \|f(x_1, u_0) - f(\bar{x}, \bar{u})\| + \|f(x_2, u_0) - f(\bar{x}, \bar{u})\| < 2\delta$$

where $\delta = max(\delta_1, \delta_2)$, and is arbitrary positive number. This implies that $f(x_1, u_0) = f(x_2, u_0)$, but this contradicts the assumption that $f$ is one to one.
Case 2. Otherwise, if $x_1 = x_2$, the contradiction is obtained by the assumption that $S_4^P(\bar{x}, \delta_1) \neq S_4^P(\bar{x}, \delta_2)$. So, the proposition is proved.

- **Proposition 3.8 (Uniform compactness of $S_4^P$)**

The P- The P-Stability notion map $S_4^P$ is uniformly compact at any efficient solution $\bar{x} \in \mathbb{R}^n$, if f is convex on $\mathbb{R}^n \times \mathbb{R}^m$ and $\Phi = Min_P Y(.)$ is closed.

**Proof.** To prove the proposition, it is enough to show that $S_4^P(\bar{x}, \delta)$ is closed. For this sake consider a sequence $\{u_k\} \xrightarrow[k\to\infty]{} u_0$, where $\{u_k\}$ in $S_4^P(\bar{x}, \delta)$. So there exist $x_k \in X(\bar{u})$ such that

$$\|f(x_k, u_k) - f(\bar{x}, \bar{u})\| < \delta, \qquad (7)$$

and $f(\bar{x}, u_k) \in Min_P Y(u_k)$. Since $Min_P Y(.)$ is closed and f is continuous, then $f(\bar{x}, u_0) \in Min_P Y(u_0)$ and,

$$\lim_{k\to\infty} f(x, u_k) = f(x, u_0)$$

for any $x \in X(\bar{u})$. Thus, for $k_0$ big enough, we have

$$\left\|f\left(x_{k_0}, u_{k_0}\right) - f\left(x_{k_0}, u_0\right)\right\| < \epsilon, \qquad (8)$$

where $\epsilon \ll 1$. Since $\epsilon$ is an arbitrary positive number Inequalities (7) and (8) implies that

$$\begin{aligned}\left\|f\left(x_{k_0}, u_0\right) - f(\bar{x}, \bar{u})\right\| &\leq \left\|f\left(x_{k_0}, u_{k_0}\right) - f\left(x_{k_0}, u_0\right)\right\| \\ &+ \left\|f\left(x_{k_0}, u_{k_0}\right) - f(\bar{x}, \bar{u})\right\| < \delta.\end{aligned}$$

Then $u_0 \in S_4^P(\bar{x}, \delta)$. So, the proposition is proved.

- **Proposition 3.9 (Upper semi-continuity of $S_4^P$)**

The P-Stability notion map $S_4^P$ is uniformly compact at any efficient solution $\bar{x} \in \mathbb{R}^n$, if f is convex on $\mathbb{R}^n \times \mathbb{R}^m$ and $\Phi = Min_P Y(.)$ is closed.
***Proof.*** Consider a sequence $\{\bar{x}_k\} \subset \Re^n$, $s.t. \bar{x}_k \to \bar{x}$ and $u_k \in S_4^P(\bar{x}, \delta)$ such that $u_k \to u_0$ to prove the proposition, we must show that $u_0 \in S_4^P(\bar{x}, \delta)$. Indeed, $u_k \in S_4^P(\bar{x}, \delta)$ then

$$\exists x_k \in X(\bar{u}), \qquad \|f(x_k, u_k) - f(\bar{x}, \bar{u})\| < \delta$$

and

$$\bar{y}_k = f(\bar{x}, u_k) \in Min_P Y(u_k).$$

Since $f$ is convex, then $f$ is continuous and this means that

$$\lim_{k\to\infty} \bar{y}_k = \lim_{k\to\infty} f(\bar{x}, u_k) = f(\bar{x}, u_0).$$

Since $Min_P Y(.)$ is closed. Hence, $f(\bar{x}, u_0) \in Min_P Y(u_0)$ and for any $x \in X(\bar{u})$,

$$\lim_{k\to\infty} f(x, u_k) = f\left(x, \lim_{k\to\infty} u_k\right) = f(x, u_0).$$

Therefore, for $k_0$ big enough, we have

$$\begin{aligned}\left\|f\left(x_{k_0}, u_0\right) - f(\bar{x}, \bar{u})\right\| &\leq \left\| f\left(x_{k_0}, u_0\right) - f\left(x_{k_0}, u_{k_0}\right) \right\| \\ &+ \left\| f\left(x_{k_0}, u_{k_0}\right) - f(\bar{x}, \bar{u}) \right\| < \delta.\end{aligned}$$

Hence, $\left\|f\left(x_{k_0}, u_0\right) - f(\bar{x}, \bar{u})\right\| \leq \delta$ and $f(\bar{x}, u_0) \in Min_P Y(u_0)$, then $u_0 \in S_4^P(\bar{x}, \delta)$ and the proposition is proved.

The above-stated propositions prove the following theorem on the novel P-stability set-valued maps of the first, second, and third kinds, which form the core result of this paper.

- **Theorem 3.1** The P-Stability notion maps $\{S_i^P\}_{i=1,3,4}$ are uniformly compact, and sequentially upper semi-continuous at any efficient solution $\bar{x} \in \Re^n$, if f is convex on $\mathbb{R}^n \times \mathbb{R}^m$ and $Min_P Y(.)$ is closed.

## 4. P-Perturbation Maps

Initially, we want to define the P-perturbation set-valued maps over $\mathbb{R}^n$ into $\mathcal{P}(\mathbb{R}^p)$ which are denoted by $\psi_i^P$, for $i = 1,2,3$ and 4, as follows:

- **Definition 4.1 (The P-perturbation maps $\{\psi_i^P\}_{i=1,2,3,4}$)**

Define the set-valued maps $\psi_i^P$, for $i = 1,2,3$ and 4, from $\mathbb{R}^n$ to $\mathcal{P}(\mathbb{R}^p)$ as follows:

$$\psi_i^P(\bar{x}) = \{\bar{y} \in \Re^p : \bar{y} = f(\bar{x}, \bar{u}), \bar{u} \in S_i^P(\bar{x})\}, \quad \text{for } i = 1,2,3,4.$$

For sake of clarity the compact notation $S_i^P(\bar{x})$ is used for all the $S_i^P$. Important properties for the set-valued maps $\psi_i^P$ $(i = 1,2,3,4)$ are introduced and proved in consequent. It is to be noted that, if $f$ is strictly convex on $\mathbb{R}^n \times \mathbb{R}^m$ and $\bar{x}_1, \bar{x}_2$ are different efficient solutions in $\mathbb{R}^n$; then

$$\psi_i^P(\bar{x}_1) \cap \psi_i^P(\bar{x}_2) = \emptyset, \qquad i = 1,2$$

This follows from the uniqueness of $S_i^P$ (Proposition 3.1). Similarly, if $f$ is one-to-one, and $\bar{x}_1$and $\bar{x}_2$ are different efficient solutions in $\mathbb{R}^n$, then

$$\psi_i^P(\bar{x}_1) \cap \psi_i^P(\bar{x}_2) = \emptyset, \qquad i = 3,4$$

(Proposition 3.4, Proposition 3.7).

- **Theorem 4.1 (Upper semi-continuity of $\{\psi_i^P\}_{i=1,2,3,4}$)**

If f is convex on $\mathbb{R}^n \times \mathbb{R}^m$, then the P-perturbation map $\psi_i^P$ (i = 1,3,4) is upper semi-continuous at any efficient solution $\bar{x} \in \mathbb{R}^n$.

**Proof.** Consider a sequence of efficient solutions $\{\bar{x}_k\} \subset \Re^n$ such that $\bar{x}_k \xrightarrow[k\to\infty]{} \bar{x}$ and let $\bar{y}_k \in \psi_i^P(\bar{x}_k)$ such that $\bar{y}_k \xrightarrow[k\to\infty]{} \hat{y}$. To manifest the theorem, it is enough to prove that $\hat{y} \in \psi_i^P(\bar{x})$. Assume the contrary, i.e., there exists no $u \in S_i^P(\bar{x})$, such that $\hat{y} = f(\bar{x}, u) \in \mathrm{Min}_P Y(u)$.

We have,

$$\hat{y} = \lim_{k\to\infty} \bar{y}_k = \lim_{k\to\infty} \psi_i^P(\bar{x}_k) = \lim_{k\to\infty} f(\bar{x}_k, \bar{u}_k) \text{ for } \bar{u}_k \in S_i^P(\bar{x}_k).$$

Since f is convex function, then f is continuous [27], and so

$$\hat{y} = \lim_{k\to\infty} f(\bar{x}_k, \bar{u}_k) = f\left(\lim_{k\to\infty} \bar{x}_k, \lim_{k\to\infty} \bar{u}_k\right).$$

The uniform compactness of $S_i^P(\bar{x})$ (Proposition 3.2, Proposition 3.5, Proposition 3.8), implies that $\exists \hat{u} \in S_i^P(\bar{x})$ such that $\lim_{k\to\infty} \bar{u}_k = \hat{u}$. So

$$\hat{y} = f\left(\lim_{k\to\infty} \bar{x}_k, \lim_{k\to\infty} \bar{u}_k\right) = f(\bar{x}, \hat{u}), \qquad \hat{u} \in S_i^P(\bar{x}).$$

This is contradicting with the assumption. Therefore, the theorem is proved.

The remainder of this section is dedicated to defining the P-Stability perturbation maps. For $i = 1,2,3,4$, the perturbation map is denoted by $\Theta_i^P$, and defined as the composite between the inverse of P-Stability notion $(S_i^P)^{-1}$, if it exists, and the perturbation map $\psi_i^P$ ($i = 1,2,3,4$) as follows:

$$\Theta_i^P : \mathbb{R}^m \xrightarrow{(S_i^P)^{-1}} \mathbb{R}^n \xrightarrow{\psi_i^P} \mathcal{P}(\mathbb{R}^p);$$

i.e., for $i = 1,2,3,4$

$\Theta_i^P : \mathbb{R}^m \to \mathcal{P}(\mathbb{R}^p)$

$$u \to \Theta_i^P(u) = \psi_i^P \circ (S_i^P)^{-1}(u) = \psi_i^P\left((S_i^P)^{-1}(u)\right)$$

The connection between the Egyptian and Japanese approaches stems from the definitions of $\psi_i^P$ and $S_i^P$ ($i = 1,2,3,4$).

## 5. Illustrative Example

Indeed, to obtain the $P$-stability sets $S_i^P(\bar{x})$, for a feasible solution $\bar{x}$ and $i = 1,2$, the subsequent steps can be applied

**Step 1***:* Frame the optimization problem and involve the parameters under investigation.

**Step 2**: Start with any certain $\bar{u} \in S_S^P$ and use a suitable software package to obtain efficient solution $\bar{x}$.

**Step 3:** Use any scalarization technique, for example the non-negative weighting [13, 14] to form a single objective nonlinear programming problem corresponding to $(E_P(u))$. For $w \in W = \{w \in \mathbb{R}^p : \sum_{j=1}^p w_j = 1, w_j \geq 0\}$, we get

$$E_{P,w}(u) \to \begin{cases} P - \min\limits_{x \in X(u)} \sum_{j=1}^{p} w_j\, f_j(x,u) \\ \text{subject to} \\ X(u) = \{x \in \mathbb{R}^n : g_i(x,u) \leq 0, i = 1, \dots, q\} \end{cases}$$

**Step 4:** Formulate the Kuhn-Tucker conditions (KTC) (corresponding to side I in case of $S_2^P$ ) [4, 15, 17, 28] of the problem $E_{P,w}(u)$

$$\begin{cases} \dfrac{\partial F_w(x,u)}{\partial x_j} + \sum_{r=1}^{q} \alpha_r \dfrac{\partial g_r(x,u)}{\partial x_j} = 0, & j = 1, \dots, n \\ \alpha_i g_i(x,u) = 0 \text{ and } \alpha_i \geq 0\,, & i = 1, \dots, q \end{cases}$$

where

$$F_w(x,u) = \sum_{j=1}^{p} w_j\, f_j(x,u)$$

**Step 5:** Get the set of all $P$-minimal points $Min_P Y(u)$.

**Step 6:** The $P$-stability set $S_i^P(\bar{x})$ is collected as the intersection between the unions of the sets corresponding to the possible values of the multipliers as in [24, 25] and the set of parameters with optimum in $Min_P Y(u)$. Note that, this last set lies in the domination cone $P$.

Consider the following illustrative example, for $u \in \Re$,

$$E_P(u) \to \begin{cases} P - \min\limits_{x \in X(u)} f(x,u) = (ux_1, ux_2) \\ \text{subject to} \\ X(u) = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 \geq u, x_1 \leq u, x_2 \leq u\} \subset \mathbb{R}^2 \end{cases}$$

where $P$ is the top half of the first quadrant, i.e.

$$P = \{(y_1, y_2) \in \mathbb{R}_+^2 : y_2 \geq y_1\}.$$

We have

$$Y(u) = \{y \in \mathbb{R}^2 : y = f(x,u), x \in X(u)\}$$
$$= \{y = (y_1, y_2) \in \mathbb{R}^2 : y_1 + y_2 \geq u^2, y_1 \leq u^2, y_2 \leq u^2\},$$

and

$$\begin{aligned}\Phi(u) &= \mathrm{Min}_P Y(u) \\ &= \left\{(y_1, y_2) \in Y(u): \big(Y(u) - (y_1, y_2)\big) \cap (-P) = \{0\}\right\} \\ &= \{(y_1, y_2) \in Y(u): y_2 = -y_1 + u^2, 0 \le y_1 \le y_2 \le u^2\}.\end{aligned}$$

The solvability set is then

$$S_S^P = \{u \in \mathbb{R}: u = x_1 + x_2:\ (x_1, x_2) \in X(u), 0 \le ux_1 \le ux_2 \le u^2\}.$$

The Lagrange operator for the weighted single valued objective function is

$$\begin{aligned}L(x_1, x_2, \alpha_1, \alpha_2, \alpha_3, u) &= wux_1 + (1-w)ux_2 \\ &\quad - \alpha_1(x_1 + x_2 - u) + \alpha_2(x_1 - u) \\ &\quad + \alpha_3(x_2 - u),\end{aligned}$$

where $0 \le w \le 1$, and the Kuhn-Tucker conditions are

$$K(x, u) = 0,$$

where

$$K(x, u) = \begin{bmatrix} wu - \alpha_1 + \alpha_2 \\ (1-w)u - \alpha_1 + \alpha_3 \\ -\alpha_1(x_1 + x_2 - u) \\ \alpha_2(x_1 - u) \\ \alpha_3(x_2 - u) \end{bmatrix}$$

To obtain the first kind stability set $S_1^P\big((0,1)\big)$, Steps 1 through 5 are applied. In fact, for the KTC to be valid at the parameter $\bar{u} = 1$ with feasible solution $\bar{x} = (0,1) \in X(1)$, then $\alpha_2 = 0$. Hence,

$$wu - \alpha_1 = 0$$
$$(1-w)u - \alpha_1 + \alpha_3 = 0$$

Then the first kind stability set is

$$S_1^P\big((0,1)\big) = \{u \in S_S^P: u = 2\alpha_1 - \alpha_3, 2\alpha_1 \ge \alpha_3 \ge 0\}.$$

Moreover, the second kind stability set $S_2^P(\bar{x}, \{1\})$ is obtained in an analogous manner by applying Steps 1 through 5, but with KTC restricted to $I$. The KTC restricted to the linear side $g_1(x)$ are

$$wu - \alpha_1 = 0$$
$$(1-w)u - \alpha_1 = 0$$
$$\alpha_1(x_1 + x_2 - u) = 0$$

Hence,

$$S_2^P(\bar{x}, \{1\}) = \left\{u \in S_S^P: u = 2\alpha_1, \alpha_1 \ge \frac{1}{2}\right\}.$$

To obtain the first kind stability set $S_1^P\big((0,1)\big)$, Steps 1 through 5 are applied. In fact, for the KTC to be valid at the parameter $\bar{u} = 1$ with feasible solution $\bar{x} = (0,1) \in X(1)$, then $\alpha_2 = 0$. Hence,

$$wu - \alpha_1 = 0$$
$$(1-w)u - \alpha_1 + \alpha_3 = 0$$

Then the first kind stability set is

$$S_1^P\big((0,1)\big) = \{u \in S_S^P: u = 2\alpha_1 - \alpha_3, 2\alpha_1 \ge \alpha_3 \ge 0\}.$$

Moreover, the second kind stability set $S_2^P(\bar{x}, \{1\})$ is obtained in an analogous manner by applying Steps 1 through 5, but with KTC restricted to $I$. The KTC restricted to the linear side $g_1(x)$ are

$$wu - \alpha_1 = 0$$
$$(1-w)u - \alpha_1 = 0$$
$$\alpha_1(x_1 + x_2 - u) = 0$$

Hence,

$$S_2^P(\bar{x}, \{1\}) = \left\{u \in S_S^P: u = 2\alpha_1, \alpha_1 \ge \frac{1}{2}\right\}.$$

The third kind P-stability set $S_3^P(\bar{x}, x^*, \delta)$, for a parameter $\bar{u} \in S_S^P$, with efficient solution $\bar{x} \in X(\bar{u})$, and feasible point $x^* \in X(\bar{u})$, is obtained as the intersection between the set of parameters $u$ with optimum in $Min_P Y(u)$ and those with image, under the objective function, is in the neighborhood of $f(\bar{x}, \bar{u})$.

Consider the parameter $\bar{u} = 1 \in S_S^P$, with feasible solution $\bar{x} = (0,1) \in X(1)$. Let $x^* = (1,1) \in X(1)$ and $\delta > 0$, then $f\big((0,1), 1\big) = (0,1)$ and $f\big((1,1), u\big) = (u, u)$. As $f((0,1), u) \in Min_P Y(u)$, then $u \ge 1$. For $\|(y_1, y_2)\| = w|y_1| + (1-w)|y_2|$ a norm on $\mathbb{R}^2$, with $0 \le w \le 1$, the third kind P-stability set is

$$S_3^P\big((0,1), (1,1), \delta\big) = \{u \in S_S^P: 1 \le u \le (1+\delta) - w\}.$$

Note that, if $\delta < w$, then $S_3^P\big((0,1), (1,1), \delta\big) = \emptyset$, for the chosen norm. Lastly, the fourth kind P-stability set is

$$S_4^P\big((0,1), \delta\big) = \{u \in S_S^P: \exists (x_1, x_2) \in X(u), u \ge 1, w(ux_1) + (1-w)|1 - ux_2| < \delta\}.$$

## 6. Conclusion

In a parametric convex vector optimization problem, stability and sensitivity analysis are critical. This stems from the various applications of these problems. In this study, we address a parametric vector optimization problem over a convex cone P. The P-Stability notion sets from first through fourth kind for such optimization problems are introduced together with their associated notion maps. Among qualitative mathematical properties, uniqueness, uniform compactness, and sequential semi-continuity are checked and proved for the notion maps corresponding to the P-stability notion sets of the first, third, and fourth kinds. Moreover, the perturbation maps accompanying the stability notion maps are defined, and the uniqueness and sequential semi-continuity are shown wherever achievable. For future work, the same arguments might be extended to further stability maps with higher orders. Besides, the quantitative properties of stability and perturbation maps can be studied for generalized conic optimization problems.

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