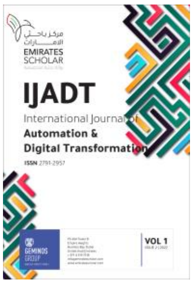

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Measurement of Angle Between Subspaces in Direct Sum Decomposition

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Abstract
The problems of Eigen structure assignment has been studied. As an immediate part of that, the problem addressed is the development of measures of “skewness” between subspaces which are in the direct sum decomposition of the state space and a concept of angle between a set of subspaces.

Keywords:
Angle between spaces, Condition Number, Gramian, Skewness of eigen frames, Spread of singular values.

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1. Introduction

The problem that frequently emerges in the study of performances of linear systems is the issue of *"skewness"* of eigenframes. This problem is linked to sensitivity of eigenvalues to parameter uncertainty, perturbations, as well as sensitivity of Nyquist diagrams to model parameter uncertainty.

These skewness properties are also linked to measures of controllability and observability, when these are assessed in their model setting. So far, the measure of skewness has been considered on eigenframes corresponding to distinct eigenvalues and thus standard tools such as the Gramian, Singular Value Decomposition, Condition Number, Sdur compliment can be used. However, frequently, we have eigenframes corresponding to repeated eigenvalues, complex eigenvalues, where a vector basis set is not uniquely defined, although the corresponding subspaces are.

The problem that is addressed here is the development of measures of *"skewness"* between subspaces defining a direct sum decomposition of the state space and thus developing a concept of angle between sets of subspaces.

The aim of the paper is to provide the required new concept of the relative positioning between subspaces that can be used in quantifying:

- Sensitivity of eigenvalues
- Relative measures of controllability and observability.
- Deviations from strong stability to overshooting behaviour.

This work is based on:

1. Development of general properties for positioning of subspaces in direct sum decomposition.
2. Development of measures of skewness using:
    - The Gramian
    - Condition number
    - Spread of singular values

Our intention in this paper is to produce some results which could provide the bases for the computation of the most orthogonal decomposition of the state space into controllability spaces. This is considered as a first step in selecting a set of closed-loop eigenvectors which are nearly orthogonal and thus achieve reduced sensitivity. This discussion involves parametrising the family of controllability subspaces using results on the parameterisation of minimal bases. The solution to the problem of finding the most orthogonal decomposition still remains open.

2. Problem statement and preliminary results

Let us consider the direct sum decomposition of $\mathbb{R}^n$ in terms of some spaces V_i such that $V_i \in \mathbb{R}^n, \dim V_i = \rho_i, i = 1,2, \dots, k$. i.e. $\mathbb{R}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_k$. (2.1)

The set of such spaces $\{V_i, i \in \{1,2,\cdots,k\}\}$ will be referred to as a decomposing set of $\mathbb{R}^n$. Clearly, these spaces are linearly independent. What we want to investigate is the relative "degree" of independency between these spaces.

The spaces V_i are assumed given and may represent the generalised eigenspaces associated with repeated eigenvalues, or the two-dimensional space associated with a pair of complex conjugate eigenvalues, or the higher order spaces associated with repeated complex eigenvalues.

Let V be a basis of $\mathbb{R}^n$ defined as:

$$V = [\underline{V_1}|\underline{V_2}|\cdots|\underline{V_k}]$$
(2.2)

where $\underline{V_1}$ is a basis of V_i. We can always assume that the columns of V_i are normalised to unit length. Clearly, $V_i \in \mathbb{R}^{n \times \rho_i}$ and so for any square matrix Q such that:

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$$Q = \begin{bmatrix} Q_1 & 0 & \cdots & 0 & 0 \\ 0 & Q_2 & 0 & \vdots & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & Q_{k-1} & 0 \\ 0 & \cdots & 0 & 0 & Q_k \end{bmatrix}, \quad Q_i \in \mathbb{R}^{\rho_i \times \rho_i} \text{ and}$$

$|Q_i| \neq 0, i = 1,2,\cdots,k,$

any other basis of $\mathbb{R}^n$, consistent with the (2.1) decomposition is given by:

$$\tilde{V} = [\underline{\tilde{V}_1}|\underline{\tilde{V}_2}|\cdots|\underline{\tilde{V}_k}] = [\underline{V_1}|\underline{V_2}|\cdots|\underline{V_k}] \begin{bmatrix} Q_1 & 0 & \cdots & 0 & 0 \\ 0 & Q_2 & 0 & \vdots & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & Q_{k-1} & 0 \\ 0 & \cdots & 0 & 0 & Q_k \end{bmatrix}. \tag{2.3}$$

Bases such as those defined above, will be referred to as $\{V_i\}_k -$ structured bases of $\mathbb{R}^n$. Of special interest are the so called normal $- \{V_i\}_k -$ structured bases which are defined by the property that the columns of each $V_i \in \mathbb{R}^n$ are orthonormal, i.e.

$$V_i{}^t V_i = I_{\rho_i}, \quad i = 1,2,\cdots,k \tag{2.4}$$

Normal $- \{V_i\}_k -$ structured bases are examined first. One may preliminarily compare the structure of singular values corresponding to any two normal $- \{V_i\}_k -$ structured bases as follows:

Proposition 1. Let V_i and $\tilde{V}_i$ be two normal $- \{V_i\}_k -$ structured bases, then V_i and $\tilde{V}_i$ have the same singular values.

Proof: If V and $\tilde{V}$ are normal $- \{V_i\}_k -$ structured bases, then they are related as:

$$\tilde{V} = [\underline{\tilde{V}_1}|\underline{\tilde{V}_2}|\cdots|\underline{\tilde{V}_k}] = [\underline{V_1}|\underline{V_2}|\cdots|\underline{V_k}] \begin{bmatrix} Q_1 & 0 & \cdots & 0 & 0 \\ 0 & Q_2 & 0 & \vdots & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & Q_{k-1} & 0 \\ 0 & \cdots & 0 & 0 & Q_k \end{bmatrix}$$

where Q_i are orthogonal i.e., $Q_i^t Q_i = I_\rho$, $i = 1,2,\cdots,k$.

Clearly, $\tilde{V}^t \tilde{V} = diag\{Q_i^t\}.diag\{Q_i\}$ and since $diag\{Q_i\}$ are orthogonal, then V and $\tilde{V}$ have the same singular values.

As we expected the above result suggests that any selection of orthogonal bases leads to the same singular values (all equal to one).

However, the main question arises when one of the bases is not necessarily orthogonal. We will investigate this as follows:

Example 1: Given the direct sum decomposition as in (2.1), where $dim(V) = e_i$,

$V_i = [\underline{v}_{1i}, \underline{v}_{2i}, \cdots, \underline{v}_{e_i i}]$ an orthonormal basis of V_i generates alternative bases for V_i,

$\tilde{V}_i = [\underline{\tilde{v}}_{1i}, \underline{\tilde{v}}_{2i}, \cdots, \underline{\tilde{v}}_{e_i i}]$, not necessarily orthonormal such that

$\|\underline{\tilde{v}}_{ji}\| = 1, i = 1,2,\cdots,k, \; j = 1,2,\cdots,e_i.$

Proposition 2: If $V_i = [\underline{v}_{1i}, \underline{v}_{2i}, \cdots, \underline{v}_{e_i i}]$, $i = 1,2,\cdots,k$ are orthonormal bases of V_i, then $\tilde{V}_i = [\underline{\tilde{v}}_{1i}, \underline{\tilde{v}}_{2i}, \cdots, \underline{\tilde{v}}_{e_i i}]$ is also a basis with $\|\underline{\tilde{v}}_{ji}\| = 1$, if and only if $\tilde{V}_i = V_i Q_i$, $Q_i = [\underline{q}_{1i}, \underline{q}_{2i}, \cdots, \underline{q}_{e_i i}]$, in which $\|\underline{q}_{ji}\| = 1, i = 1,2,\cdots,k, \; j = 1,2,\cdots,e_i.$

Proof: $\tilde{V}_i$ *and* V_i are linked as:

$$\tilde{V}_i = [\underline{\tilde{v}}_{1i}, \underline{\tilde{v}}_{2i}, \cdots, \underline{\tilde{v}}_{e_i i}] = [\underline{v}_{1i}, \underline{v}_{2i}, \cdots, \underline{v}_{e_i i}] \begin{bmatrix} q_{1i}^1 & \cdots & q_{e_i i}^1 \\ \vdots & & \vdots \\ q_{1i}^{e_i} & \cdots & q_{e_i i}^{e_i} \end{bmatrix}$$

Hence, $\underline{\tilde{v}}_{ji} = \underline{v}_{1i} q_{ji}^1 + \underline{v}_{2i} q_{ji}^2 + \cdots + \underline{v}_{e_i i} q_{ji}^{e_i}$, $j = 1,2,\cdots,e_i$, and

$$\|\tilde{v}_{ji}\|^2 = \underline{\tilde{v}}_{ji}{}^T \underline{\tilde{v}}_{ji} = \left(\underline{v}_{1i} q_{j1}^i + \cdots + \underline{v}_{e_i i} q_{ji}^{e_i}\right)^T \left(\underline{v}_{1i} q_{j1}^i + \cdots + \underline{v}_{e_i i} q_{ji}^{e_i}\right) = \left(q_{ji}^i\right)^2 + \cdots + \left(q_{ji}^{e_i}\right)^2 = \|q_{ji}\|^2, j = 1,2,\cdots,e_i$$

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due to orthogonality, and hence, $\|\underline{\tilde{v}}_{ji}\| = 1$ if and only if $\|\underline{q}_{ji}\| = 1$.

3. Measuring the degree of orthogonality

In this part and based on the above result, we will use different type metrics to define the degree of orthogonality of the decomposition, or alternatively to measure the skewness of the direct sum decomposition.

3.1. The Gramian

A standard test for checking the degree of orthogonality is that based on the *volume* or the *Gramian* and so, the Gramian of the $\tilde{V}$ matrix, in (2.3), is given by:

$$G(\tilde{V}) = \det(\tilde{V}^t\tilde{V}) = \left|diag\{Q_i^t\}\det(\tilde{V}^t\tilde{V})diag\{Q_i\}\right| \quad (3.1)$$

and since V is orthogonal with unit length, then

$$G(\tilde{V}) = \det(V^tV).\ \det(diag\{Q_i^t\}.\ diag\{Q_i\}) \quad (3.2)$$

According to the *Hadamard's inequality theorem* [1], [2], the determinant of a matrix, when it's restricted to real numbers, can be bounded in terms of the lengths of its vectors. Specifically, Hadamard's inequality states that if N is the matrix having columns

$$\underline{v}_i,\ i = 1,2,\cdots,n, \quad \text{then} \quad |\det(N)| \le \prod_{i=1}^{n}\|\underline{v}_i\| \quad (3.3)$$

Clearly, in our case, since the length of the vectors belong to $G(\tilde{V})$ can be in the range from 0 to 1. So, as a result, $\det\left(G(\tilde{V})\right)$ will also be in the range from 0 to 1.

The main objective to be studied is the condition in which this value is maximum or in other hand, the vectors in $\tilde{V}$, has maximum angle.

Proposition 3: If $V = [\underline{V_1}|\underline{V_2}|\cdots|\underline{V_k}]$ is any basis corresponding to $\mathbb{R}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_k$ decomposition where V_i is an orthogonal basis of V with unit length vectors, then:

(i) The singular values of V^tV are invariant of any selection of orthogonal basis.

(ii) The value of $\det(V^tV)$ is invariant of any selection of the orthogonal basis.

Proof: Any two orthonormal bases $V, \tilde{V}$ are related by (3.1) as:

$$\tilde{V} = [\underline{\tilde{V}_1}|\underline{\tilde{V}_2}|\cdots|\underline{\tilde{V}_k}]$$
$$= [\underline{V_1}|\underline{V_2}|\cdots|\underline{V_k}]\begin{bmatrix} Q_1 & 0 & \cdots & 0 & 0 \\ 0 & Q_2 & 0 & \vdots & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & Q_{k-1} & 0 \\ 0 & \cdots & 0 & 0 & Q_k \end{bmatrix}$$

where Q_i are orthogonal bases, i.e., $Q_i^tQ_i = I_{\rho_i}$. Thus $V^tV = diag\{Q_i^t\}\ .V^tV.\ diag\{Q_i\}$,

and since Q_i are orthogonal, the result follows.

Assume now that the $\{V_i, i \in \{1,2,\dots,k\}\}$ bases are orthogonal, and we select another arbitrary bases $\tilde{V}_i = V_iQ_i$ with unit length vectors, but not necessarily orthogonal. Inspection of equation (2.3) and the latest result suggest that the value of $G(\tilde{V})$ really depends on the property of the matrix T where T is as follows:

$$T = \begin{bmatrix} Q_1^t & 0 & \cdots & 0 & 0 \\ 0 & Q_2^t & 0 & \vdots & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & Q_{k-1}^t & 0 \\ 0 & \cdots & 0 & 0 & Q_k^t \end{bmatrix}\begin{bmatrix} Q_1 & 0 & \cdots & 0 & 0 \\ 0 & Q_2 & 0 & \vdots & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & Q_{k-1} & 0 \\ 0 & \cdots & 0 & 0 & Q_k \end{bmatrix}$$
$$= diag\{Q_i^tQ_i\} \quad (3.4)$$

or its determinant $|T|$ defined as $|T| = \prod_{i=1}^{k}|Q_i^tQ_i|$.

For any matrix $Q \in \mathbb{R}^{\rho_i \times \rho_i}$, $Q_i = [\underline{q}_1, \underline{q}_2, \cdots, \underline{q}_{\rho_i}]$ with $|\underline{q}_i| = 1$, we have that:

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$$p = Q^t Q = \begin{bmatrix} \underline{q}_1^t \\ \underline{q}_2^t \\ \vdots \\ \underline{q}_k^t \end{bmatrix} [\underline{q}_1 \quad \underline{q}_2 \quad \cdots \quad \underline{q}_k] = \begin{bmatrix} \underline{q}_1^t\underline{q}_1 & \underline{q}_1^t\underline{q}_2 & \cdots & \underline{q}_1^t\underline{q}_k \\ \underline{q}_2^t\underline{q}_1 & \underline{q}_2^t\underline{q}_2 & \cdots & \underline{q}_2^t\underline{q}_k \\ \vdots & \vdots & \vdots & \vdots \\ \underline{q}_k^t\underline{q}_1 & \underline{q}_k^t\underline{q}_2 & \cdots & \underline{q}_k^t\underline{q}_k \end{bmatrix} \tag{3.5}$$

or

$$P = \begin{bmatrix} 1 & \underline{q}_1^t\underline{q}_2 & \cdots & \underline{q}_1^t\underline{q}_k \\ \underline{q}_2^t\underline{q}_1 & 1 & \cdots & \underline{q}_2^t\underline{q}_k \\ \vdots & \vdots & \vdots & \vdots \\ \underline{q}_k^t\underline{q}_1 & \underline{q}_k^t\underline{q}_2 & \cdots & 1 \end{bmatrix}.$$

(3.6)

Note that the matrix P is positive definite. Furthermore:

$$|T| = \sum_{i=1}^{k} \det\{Q_i^t Q_i\}$$

The main issue is now the properties of the $\det\{Q_i^t Q_i\}$ and the investigations of the conditions under which we can maximise $\det\{T\}$. We note first the following lemma.

Lemma 1: For any $n \times n$ positive definite matrix X, with constant trace $tr[X] = \alpha$, the determinant is maximised when $X = \frac{\alpha}{n} I_n$.

Proof: Applying *Hadamard inequality* (3.3), the determinant of an $n \times n$ matrix X is maximized when the matrix is diagonal, that is, eigenvalues of the matrix are the diagonal elements.

If $\underline{a} = (a_1, a_2, \cdots, a_n), \sum_i a_i$, is the vector of eigenvalues of X, from majorization theory [3], the vector $a^t = \left(\frac{\alpha}{n}, \frac{\alpha}{n}, \cdots, \frac{\alpha}{n}\right)$, with all elements equal, is majorized by any other vector $\underline{a}$.

Also, a majorization result says that if g is a continues nonnegative function on $I \subset \mathbb{R}$, a function $\varphi(X) = \prod_{i=1}^{n} g(x_i)$ is Schur-concave (convex) on I^n, if and only if $log(g) \ln g$ is concave (convex) in I^n. In our case, *log(x)* is a concave function on $\mathbb{R}^+$and $\det(X) = \prod_{i=1}^{n} a_i$ is a Schur-concave function and its maximum is attained for a^t. Having all eigenvalues equal is equivalent to saying that X is a scaled identity matrix, under its trace constraint [4].

For our case, the matrix P which has: $trace[P] = k$, will have its determinant maximised when $P = I$, i.e. the transformation Q_i are orthogonal. This then leads to the following main result.

Theorem 1: Let us consider the decomposition of $\mathbb{R}^n$ as:

$$\mathbb{R}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_k, \qquad \dim V_i = \rho_i,\ i = 1,2,\ldots,k.$$

and let V_i be a basis for each of the V_i spaces of vectors with unit length. Then the Gramian of the basis $V = [\underline{V_1}|\underline{V_2}|\cdots|\underline{V_k}]$ is:

$$G(V) = det(\begin{bmatrix} \underline{V_1^t} \\ \underline{V_2^t} \\ \vdots \\ \underline{V_k^t} \end{bmatrix} V = [\underline{V_1}|\underline{V_2}|\cdots|\underline{V_k}]) \tag{3.7}$$

and it is maximised if and only if the bases V_i for the V_i subspaces are orthogonal with unit length.

Proof: The invariant of $G(V)$ for the selection of different bases has been
established.
This together with *lemma 1*, establishes the result.

The above establishes $G(V)$, where $\{V_i\}$ are any orthogonal, unit length, as a measure of the angle between a set of subspaces, that will be defined as the *Gramian angle* of the $\{V_i, i \in \{1,2,\ldots,k\}\}$ decomposition.

3.2. Condition Number

The Condition Number could be considered to be used as another measurement tool in order to measure the "*skewness*" of eigenframes.

The condition number is defined as: $\kappa(A) = \frac{\sigma_{\max}(A)}{\sigma_{\min}(A)}$

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where σ_{max} and σ_{min} are maximal and minimal singular values of A respectively.

In general: $\kappa(A) \geq 1$, and hence for any normal matrix $\kappa(A) = 1$ as all the singular values of the normal matrix are equal to 1.

Considering the above description of condition number, we will define another measure of the degree of orthogonality or another measure of skewness of the decomposition.

Definition: If $A \in M^{m\times n}$ is a given matrix, and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_q \geq 0$, $q = \min(m,n)$, then $\|A\| = \sigma_1$, [5].

Lemma 2: For any matrix $A \in F^{m\times n}$, there are n singular values such that:

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_q > 0$; where $\sigma_{\max} = \sigma_1$ and $\sigma_{\min} = \sigma_n$, [5].

Corollary 1: Let $A \in F^{m\times n}$ and $A \in F^{m\times n}$, then for all $i = 1,2,\dots,\min(m,n)$, we will have

$\sigma_i(A)\,\sigma_{\min}(B) \leq \sigma_i(AB) \leq \sigma_i(A)\,\sigma_{\max}(B)$ [6].

Corollary 2: Let $A \in F^{m\times n}$. If $n = m$ and A is non-singular, then:

$\|A^{-1}\| = \sigma_{\min}(A^{-1}) = \frac{1}{\sigma_{\max}(A)}$ [6].

Proposition 4: We consider the direct sum decomposition on $\mathbb{R}^n$, which is:

$\mathbb{R}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_k$,
(3.8)

where $\dim(V_i) = \rho_i$, i =1,2, ..., k, and all bases $V_i \in \mathrm{V}_i$ in the decomposition (3.8) to have unit length vectors. Then the Condition Number of the basis $V = [\underline{V_1}|\underline{V_2}|\cdots|\underline{V_k}]$ is

$$\kappa(V) = \frac{\sigma_{\max}(V)}{\sigma_{\min}(V)}$$

and it is minimized if and only if the bases V_i for the V_i subspaces are orthonormal.

Proof: Let the columns of V_i form an orthonormal basis of V_i. Then all other bases of V_i consisting of vectors of unit length are given as V_iQ_i where $\det(Q_i) \neq 0$ and all the columns of Q_i have unit length. Thus, all bases of $\mathbb{R}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_k$ can be written as

$$\tilde{V} = [\underline{\tilde{V}_1}|\underline{\tilde{V}_2}|\cdots|\underline{\tilde{V}_k}] = [\underline{V_1}|\underline{V_2}|\cdots|\underline{V_k}]\begin{bmatrix} Q_1 & 0 & \cdots & 0 & 0 \\ 0 & Q_2 & 0 & \vdots & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & Q_{k-1} & 0 \\ 0 & \cdots & 0 & 0 & Q_k \end{bmatrix} V.Q;\ Q \in \Phi$$

where Φ is defined as the set of all block diagonal matrices $Q = \mathrm{diag}(Q_1,\dots,Q_k)$, such that $\det(Q_i) \neq 0$ and all the columns of Q_i have unit length.

We should show that $\min_{Q\in\Phi} \kappa(VQ) = \kappa(V)$ and that, the minimum is attained for $Q = \mathrm{diag}(Q_1,\dots,Q_k)$ with all Q_i orthogonal. First by using corollary 1, we have:

$\sigma_i(V)\,\sigma_{\min}(Q) \leq \sigma_i(VQ) \leq \sigma_i(V)\,\sigma_{\max}(Q)$.
(3.9)

Hence and based on *Lemma 2* and *Definition 1*, we can have from (3.11) that for $i = 1$:

$\frac{\|V\|}{\max_{j\in\underline{k}}\|Q_j^{-1}\|} \leq \|VQ\| \leq \|V\| \max_{j\in\underline{k}}\|Q_j\|$, $\underline{k} = \{1,2,\dots,k\}$
(3.10)

Now if the minimum singular value named as σ_m, then for $i = n$

$\frac{\sigma_n(V)}{\max_{j\in\underline{k}}\|Q_j^{-1}\|} \leq \sigma_n(VQ) \leq \sigma_n(V) \max_{j\in\underline{k}}\|Q_j\|$.
(3.11)

In order to obtain the condition number of $\tilde{V}$ or equally, VQ, we have:

$\kappa(\tilde{V}) = \kappa(VQ) = \frac{\sigma_{\max}(VQ)}{\sigma_{\min}(VQ)} = \frac{\|VQ\|}{\sigma_n(VQ)}$.

So, from (3.11) and (3.13), we have:

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$$\frac{\frac{\|V\|}{\max_{j\in \underline{k}}\|Q_j^{-1}\|}}{\sigma_n(V)\max_{j\in \underline{k}}\|Q_j\|} \le \kappa(VQ) \le \frac{\|V\|\max_{j\in \underline{k}}\|Q_j\|}{\frac{\sigma_n(V)}{\max_{j\in \underline{k}}\|Q_j^{-1}\|}},$$
(3.12)

or equivalently, $\frac{\kappa(V)}{\max_{j\in \underline{k}}\|Q_j\| \cdot \max_{j\in \underline{k}}\|Q_j^{-1}\|} \le \kappa(VQ) \le \kappa(V).\max_{j\in \underline{k}}\|Q_j\| \cdot \max_{j\in \underline{k}}\|Q_j^{-1}\|$ (3.13)

Note that: $\|Q_j^{-1}\| = \sigma_{n_j}^{-1}(Q_j) = \frac{1}{\sigma_{n_j}(Q_j)}$

$$\to \max_{j\in \underline{k}}\|Q_j^{-1}\| = \max_{j\in \underline{k}} \frac{1}{\sigma_{n_j}(Q_j)} = \frac{1}{\min_{j\in \underline{k}} \sigma_{n_j}(Q_j)},$$

and that (3.12) is equivalent to:

$$\frac{\kappa(V)}{\sigma(Q)} \le \kappa(VQ) \le \kappa(V)\,.\,\sigma(Q),$$
(3.14)

where: $\sigma(Q) = \frac{\max_{j\in \underline{k}}\|Q_j\|}{\min_{j\in \underline{k}} \sigma_{n_j}(Q_j)} \ge 1.$

(3.15)

From (3.14) we get: $\frac{1}{\sigma(Q)} \le \frac{\kappa(VQ)}{\kappa(V)}$ for every $Q \in \Phi$. (3.16)

Also, from (3.15) we have:
$$\min_{Q\in\Phi} \sigma(Q) = \frac{1}{\max_{Q\in\Phi} \sigma(Q)} = 1.$$
(3.17)

So, (3.19) and (3.18) lead to: $\min_{Q\in\Phi} \frac{\kappa(VQ)}{\kappa(V)} \ge 1.$ (3.18)

Using (3.16) and noting that $\sigma(Q) = 1$ if and only if $Q = \mathrm{diag}(Q_1, \dots, Q_k)$ with Q_k orthogonal [see *Lemma 2* for proof], we have $\min_{Q\in\Phi} \frac{\kappa(VQ)}{\kappa(V)} = 1$. Since the condition number of V is fixed and assumed to be minimum $(= 1)$, which means that the condition number of $\tilde{V}$ is minimum if and only if for all $Q \in \Phi$, Qs are orthonormal.

Theorem 2: Let us consider the decomposition of $\mathbb{R}^n$ as:

$$\mathbb{R}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_k$$
(3.19)

$\dim(V_i) = \rho_i, i = 1,2,\cdots,k$ and every bases V_i in the decomposition (3.19) to have unit length vectors. Then the Condition number of the basis $V = [\underline{V_1}|\underline{V_2}|\cdots|\underline{V_k}]$ is $\kappa(V) = \frac{\sigma_{\max}(V)}{\sigma_{\min}(V)}$, and it is minimized if and only if the bases V_i are orthonormal.

Proof: The invariance of $\kappa(V)$, for the selection of different bases has been established. This, together with *Lemma 2*, *Corollaries 1* and *2*, establish the result.

The above establishes $\kappa(V)$ where $\{\underline{V_i}\}$ are of orthogonal unit length, also as a measure of the angle between a set of subspaces, defined as the *Condition Number* of the $\{V_i, i \in k\}$ decomposition.

3.3 The Spread of Singular Values

So far, we have seen two different tools in orType equation here.der to measure the degree of orthogonality or to measure the skewness of the decomposition.

Another way to measure the skewness of the decomposition is to use so called "the spread of singular values of a space". Note that, by "spread of singular values", we mean the difference between the values of singular values of any decomposition. What we are interested in, is to show that the spread of singular values of decomposition is minimized when the space is orthonormal.

Example 2: Let us consider the direct sum decomposition of $\mathbb{R}^n$ in terms of subspaces $\boldsymbol{\Upsilon}_i$ and $\boldsymbol{V}_i \in \mathbb{R}^n$, $\dim(\boldsymbol{\Upsilon}_i) = \rho_i$, $i = 1,2,\cdots,k$, i.e.

$$\mathbb{R}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_k$$
(3.20)

If $V = [\underline{V_1}|\underline{V_2}|\cdots|\underline{V_k}]$ is a normal-$\{V_i\}_k$-structured bases of $\mathbb{R}^n$ and for any $V_i = [v_{1,i}, \dots, v_{\rho_i,i}]$, then any other is expressed as

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$$\tilde{V} = [\underline{\tilde{V}_1}|\underline{\tilde{V}_2}|\cdots|\underline{\tilde{V}_k}] = [\underline{V_1}|\underline{V_2}|\cdots|\underline{V_k}]\begin{bmatrix} Q_1 & 0 & \cdots & 0 & 0 \\ 0 & Q_2 & 0 & \vdots & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & Q_{k-1} & 0 \\ 0 & \cdots & 0 & 0 & Q_k \end{bmatrix} = V.Q\,;Q \in \Phi$$

where Φ is defined as the set of all block diagonal matrices $Q = \operatorname{diag}(Q_1, \dots, Q_k)$, such that $\det(Q_i) \neq 0$ and all the columns of Q_i have unit length. Here, we want to show that for all the singular values of $\tilde{V}$, we have $\sigma_i(\tilde{V}) = \sigma_i(V) = 1,\ i = 1,2,\cdots,k$, if and only if $\tilde{V}$ are normal-$\{V_i\}_k$-structured bases.

Proof: Let $V_i \in \mathbb{R}^{n\times 2}$, then $V = [\underline{V_1}|\underline{V_2}]$, $\underline{V_i} = [v_{1,i}, \cdots, v_{\rho_i,i}], i = 1,2.$

Since V is a orthonormal bases, then $\|v_{\rho_i,i}\| = 1\,, i = 1,2$, and V has full column rank and for any θ, $V = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

Since V has orthogonal columns hence $\sigma_1(\tilde{V})$=$\sigma_2(V) = 1$. Now for any other bases $\tilde{V} \in \mathbb{R}^{n\times 2}$ we have $\tilde{V} = VQ$, where $|Q| \neq 0$ and Q is a square matrix. That is, $Q = [\underline{q_1} \quad \underline{q_2}]$ and

$\|\underline{q_i}\| = 1, i = 1,2$, but Q is not necessarily orthogonal. Based on these specifications, let's choose Q as follows:

$$Q = \begin{bmatrix} \varepsilon & \delta \\ \sqrt{1-\varepsilon^2} & \sqrt{1-\delta^2} \end{bmatrix},\ 0 \le \varepsilon \le 1,\ 0 \le \delta \le 1.$$

Then from (3.23):

$$V = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \varepsilon & \delta \\ \sqrt{1-\varepsilon^2} & \sqrt{1-\delta^2} \end{bmatrix}.$$

Now, in order to obtain the singular values of $\tilde{V}$, the following procedure can be done:

$$\sigma^2(\tilde{V}) = \sigma^2(VQ) = \lambda_i(Q^tV^tVQ) = \lambda_i(Q^tQ),\ i = 1,2 \qquad (3.21)$$

The orthogonality of V gives $\qquad V^tV = I_2.$

Then (3.21) becomes:

$$\sigma^2(\tilde{V}) = \lambda_i(Q^tQ),\ i = 1,2 \qquad (3.22)$$

Now, to obtain the eigenvalues of (Q^tQ), we have:

$$Q^tQ\begin{bmatrix} \varepsilon & \sqrt{1-\varepsilon^2} \\ \delta & \sqrt{1-\delta^2} \end{bmatrix}\begin{bmatrix} \varepsilon & \delta \\ \sqrt{1-\varepsilon^2} & \sqrt{1-\delta^2} \end{bmatrix} = \begin{bmatrix} 1 & X \\ X & 1 \end{bmatrix}$$

where $\qquad X = \varepsilon\delta + \sqrt{1-\varepsilon^2} \qquad \sqrt{1-\delta^2}.$ (3.23)

And finally: $\det[\lambda I - Q^tQ] = 0$ gives the eigenvalues of Q^tQ, or in fact the eigenvalues of $\tilde{V}$.

$$\begin{vmatrix} \lambda-1 & -X \\ -X & \lambda-1 \end{vmatrix} = 0 \Rightarrow (\lambda-1)^2 - X^2. \qquad (3.24)$$

From (3.24), the values of the two eigenvalues of Q^tQ will be $\lambda_{1,2} = \{1-X, 1+X\}$.
Obviously, based on (3.22), the singular values of $\tilde{V}$ will be $\sigma_{1,2} = \{\sqrt{1-X}, \sqrt{1+X}\}$.
Since $X \ge 0$ is always true, then $\sigma_1 = \sqrt{1-X} \ge 1$ and $\sigma_2 = \sqrt{1+X} \le 1$.
The inequalities will be changed to equalities if and only if Q is also orthogonal.
The above example simply shows that for any combinations of bases other than the orthonormal ones, some of the singular values will be greater than 1 and some others less than 1. The above result gives rise another interesting issue which is strongly relative to the above problem. That is, to find the value of minimum singular value of any $\{V_i\}_k$-structured bases chosen from (3.22) as follows;

Theorem 3: Let $\mathbb{R}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_k$ and suppose that the columns of $V_j \in \mathbb{R}^{n\times n_i}$ form an orthonormal basis of $V_i, i = 1,2,\dots,k$ so that $n = \sum_{j=1}^{k} n_j$ and hence: $V = [\underline{V_1}|\underline{V_2}|\cdots|\underline{V_k}]$ is a square invertible matrix. Then $\sigma_{min}(V) \le 1$. Furthermore, this is an equality if and only if

$V_i \perp V_j\, for\ all\ i \neq j$, so that V is an orthogonal matrix.

Proof: Assume that the singular values of V are introduced as $\sigma_1 \ge \sigma_2 \ge \cdots \sigma_n > 0$ (Since V is

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square, invertible matrix, then all the singular values are positive). A direct evaluation gives:

$$V^tV = \begin{bmatrix} \underline{V}_1^t \\ \underline{V}_2^t \\ \vdots \\ \underline{V}_k^t \end{bmatrix} [\underline{V}_1 \quad \underline{V}_2 \quad \cdots \quad \underline{V}_k]$$

$$= \begin{bmatrix} I_{n_1} & \underline{V}_1^t\underline{V}_2 & \cdots & \underline{V}_1^t\underline{V}_k \\ \underline{V}_2^t\underline{V}_1 & I_{n_2} & \cdots & \underline{V}_2^t\underline{V}_k \\ \vdots & \vdots & \vdots & \vdots \\ \underline{V}_k^t\underline{V}_1 & \underline{V}_k^t\underline{V}_2 & \cdots & I_{n_k} \end{bmatrix}$$

and hence: $trace(V^tV) = \sum_{i=1}^{n} \lambda_i(V^tV) = \sum_{i=1}^{n} \sigma_i^2(V)$

$$= \sum_{j=1}^{k} n_j = n.$$

Thus, $n\sigma_n^2(V) \le \sum_{i=1}^{n} \sigma_i^2(V) = n \ \Rightarrow \sigma_n(V) \le 1.$

Next let $\sigma_n(V) = 1$. Then we have that $n\sigma_n^2(V) = n$, and hence $n\sigma_n^2(V) = \sum_{i=1}^{n} \sigma_i^2(V)$, so that $\sigma_1(V) = \sigma_2(V) = \cdots \sigma_n(V) = 1.$ This immediately implies that V is orthogonal .Conversely if V is orthogonal then all the singular values of V (including $\sigma_n(V)$) are equal to 1.

Corollary 3: Let $V = [\underline{V}_1|\underline{V}_2|\cdots|\underline{V}_k]$ with $V_j \in \mathbb{R}^{n\times n_i}$ and $= \sum_{j=1}^{k} n_j$, be a non-singular matrix with all columns of V_j normalized to 1, $(j = 1,2, \dots, k)$. Then $\sigma_{min}(V) \le 1$; furthermore $\sigma_n(V) = 1$ if and only if V is orthogonal.

Proof: Consider that V contains n-one dimensional subspaces as follows:

$$V = \begin{Bmatrix} col_1(V_1) & \dots & col_{n_1}(V_1) & col_1(V_2) & \dots & col_{n_2}(V_2) \\ & & \dots & col_{n_k}(V_k) & & \end{Bmatrix},$$

then theorem 3 applies and hence $\sigma_{min}(V) \le 1$. Furthermore, $\sigma_n(V) = 1$ if and only if V is orthogonal.

4. Conclusion

The problem of the skewness of the eigenframes in a direct sum decomposition, has been investigated using two different kind of measurement tools; Condition number and Gramian determinant, and it has been proved that for a fixed angle between subspaces of a direct sum decomposition, the optimum values of both condition number and Gramian determinant of the whole space are obtained if and only if any individual subspace contains orthonormal vectors. It also has been shown that within this situation, the spread of the singular values of the whole space is minimized.

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